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# An extension of beta function, its statistical distribution, and associated fractional operator

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**Abstract**

Recently, various forms of extended beta function have been proposed and presented by many researchers. The principal goal of this paper is to present another expansion of beta function using Appell series and Lauricella function and examine various properties like integral representation and summation formula. Statistical distribution for the above extension of beta function has been defined, and the mean, variance, moment generating function and cumulative distribution function have been obtained. Using the newly defined extension of beta function, we build up the extension of hypergeometric and confluent hypergeometric functions and discuss their integral representations and differentiation formulas. Further, we define a new extension of Riemann–Liouville fractional operator using Appell series and Lauricella function and derive its various properties using the new extension of beta function.

**Keywords:** Extended beta function; Appell series; Lauricella function; Extended hypergeometric function; Extended confluent hypergeometric function; Statistical distribution; Riemann–Liouville fractional operator

## 1 Introduction

The classical beta function is given by [1, Eq. (16), p. 18]

$$B(\Psi_1,\Psi_2)=\int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1}\,dt=\frac{\Gamma(\Psi_1)\Gamma(\Psi_2)}{\Gamma(\Psi_1+\Psi_2)}, \tag{1}$$

where $\Re(\Psi_1),\Re(\Psi_2)>0$, $\Re$ is the real part of the function. The Gauss hypergeometric and confluent hypergeometric functions are defined as [1, Eq. (6), p. 46; Eq. (1), p. 123]

$${}_2F_1(\Psi_1,\Psi_2;\Psi_3;z)=\sum_{n=0}^{\infty}\frac{(\Psi_1)_n(\Psi_2)_n}{(\Psi_3)_n}\frac{z^n}{n!}, \tag{2}$$

where $|z|<1$, $\Psi_1,\Psi_2,\Psi_3\in\mathbb{C}$; $\Psi_3\neq 0,-1,-2,\dots$ and

$${}_1\Phi_1(\Psi_2;\Psi_3;z)=\sum_{n=0}^{\infty}\frac{(\Psi_2)_n}{(\Psi_3)_n}\frac{z^n}{n!}, \tag{3}$$

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where $|z| < 1$, $\Psi_2, \Psi_3 \in \mathbb{C}$ and $\Psi_3 \neq 0, -1, -2, \ldots$ and $(\gamma)_n$ is the Pochhammer symbol defined by [1, Eq. (1), p. 22; Eq. (3), p. 23], for $\gamma \neq 0, -1, -2, \ldots,$

$$(\gamma)_n = \begin{cases} \prod_{k=1}^{n} (\gamma + k - 1), & n \in \mathbb{N}, \\ 1, & n = 0 \end{cases} \quad \text{and} \quad (\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)}.$$

The integral representations of hypergeometric and confluent hypergeometric function are [1, Theorem.16, p. 47; Eq. (9), p. 124]

$${}_2F_1(\Psi_1, \Psi_2; \Psi_3; z) = \frac{\Gamma(\Psi_3)}{\Gamma(\Psi_2)\Gamma(\Psi_3 - \Psi_2)} \int_0^1 t^{\Psi_2 - 1}(1-t)^{\Psi_3 - \Psi_2 - 1}(1 - zt)^{-\Psi_1}\, dt, \tag{4}$$

where $\Re(\Psi_3) > \Re(\Psi_2) > 0$, $|\arg(1-z)| < \pi$, and

$${}_1\Phi_1(\Psi_2; \Psi_3; z) = \frac{\Gamma(\Psi_3)}{\Gamma(\Psi_2)\Gamma(\Psi_3 - \Psi_2)} \int_0^1 t^{\Psi_2 - 1}(1-t)^{\Psi_3 - \Psi_2 - 1} e^{zt}\, dt, \tag{5}$$

where $\Re(\Psi_3) > \Re(\Psi_2) > 0$.

Mubeen et al. [2, Eq. (2.1), p. 1552] defined the extended beta function as follows:

$$\begin{aligned} B^{\alpha,\sigma}(\Psi_1, \Psi_2; p, q) &= B^{\alpha,\sigma}_{p,q}(\Psi_1, \Psi_2) \\ &= \int_0^1 t^{\Psi_1 - 1}(1-t)^{\Psi_2 - 1}\, {}_1F_1\left(\alpha, \sigma; \frac{-p}{t}\right) {}_1F_1\left(\alpha, \sigma; \frac{-q}{(1-t)}\right) dt, \end{aligned} \tag{6}$$

where $\Re(\Psi_1), \Re(\Psi_2) > 0$, $\Re(p), \Re(q) \geq 0$, $\alpha \in \mathbb{C}$, $\sigma \neq 0, -1, -2, \ldots$.

Goswami et al. [3, Eq. (12), p. 140] defined an extension of beta function as follows:

$$B^{\gamma_1,\gamma_2}_{p,q}(\eta_1, \eta_2) = \int_0^1 t^{\eta_1 - 1}(1-t)^{\eta_2 - 1}\, {}_1F_1\left(\gamma_1, \gamma_2; \frac{-p}{t} - \frac{q}{(1-t)}\right) dt, \tag{7}$$

where $\min\{\Re(p), \Re(q)\} > 0$, $\min\{\Re(\eta_1), \Re(\eta_2)\} > 0$, $\gamma_1 \in \mathbb{C}$, and $\gamma_2 \neq 0, -1, -2, \ldots$.

In the same paper the authors also defined hypergeometric and confluent hypergeometric functions using the newly defined extended beta function [3, Eqs. (13)–(14) p. 140]:

$$F^{\gamma_1,\gamma_2}_{p,q}(\eta_1, \eta_2; \eta_3; z) = \sum_{n=0}^{\infty} \frac{B^{\gamma_1,\gamma_2}_{p,q}(\eta_2 + n, \eta_3 - \eta_2)}{B(\eta_2, \eta_3 - \eta_2)} (\eta_1)_n \frac{z^n}{n!}, \tag{8}$$

where $\Re(p), \Re(q) \geq 0$, $\Re(\eta_3) > \Re(\eta_2) > 0$, $\gamma_1 \in \mathbb{C}$, and $\gamma_2 \neq 0, -1, -2, \ldots$ and

$$\Phi^{\gamma_1,\gamma_2}_{p,q}(\eta_2; \eta_3; z) = \sum_{n=0}^{\infty} \frac{B^{\gamma_1,\gamma_2}_{p,q}(\eta_2 + n, \eta_3 - \eta_2)}{B(\eta_2, \eta_3 - \eta_2)} \frac{z^n}{n!}, \tag{9}$$

where $\Re(p), \Re(q) \geq 0$, $\Re(\eta_3) > \Re(\eta_2) > 0$, $\gamma_1 \in \mathbb{C}$, and $\gamma_2 \neq 0, -1, -2, \ldots$.

The Appell series introduced by Paul Appell (1880) is the generalization of the Gauss hypergeometric series ${}_2F_1$ of one variable to two variables.

Appell's double hypergeometric functions are defined as follows [4, Eqs. (1.4.1)–(1.4.4), p. 23]:

$$F_1\big(a_1, a_2, a_2'; a_3; u, v\big) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_2')_n}{(a_3)_{m+n}} \frac{u^m v^n}{m!n!}, \tag{10}$$

$$F_2\big(a_1,a_2,a_2';a_3,a_3';u,v\big)=\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_m(a_2')_n}{(a_3)_m(a_3')_n}\frac{u^m v^n}{m!n!}, \tag{11}$$

$$F_3\big(a_1,a_1',a_2,a_2';a_3;u,v\big)=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_1')_n(a_2)_m(a_2')_n}{(a_3)_{m+n}}\frac{u^m v^n}{m!n!}, \tag{12}$$

$$F_4\big(a_1,a_2;a_3,a_3';u,v\big)=\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}}{(a_3)_m(a_3')_n}\frac{u^m v^n}{m!n!}. \tag{13}$$

Convergence conditions for the Appell series are as follows:
- $F_2$ converges for $|u|+|v|<1$;
- $F_1$ and $F_3$ converge when $|u|<1$ and $|v|<1$;
- $F_4$ converges when $|\sqrt{u}|+|\sqrt{v}|<1$.

Lauricella(1893) defined the Lauricella functions as follows [4, Eqs. (2.1.1)–(2.1.4), p. 41]:

$$F_A^{(n)}\big(a_1,a_1',\dots,a_n';a_1'',\dots,a_n'';u_1,\dots,u_n\big)=\sum_{\xi_1,\dots,\xi_n=0}^{\infty}\frac{(a_1)_{\xi_1+\cdots+\xi_n}(a_1')_{\xi_1}\cdots(a_n')_{\xi_n}}{(a_1'')_{\xi_1}\cdots(a_n'')_{\xi_n}}\frac{u_1^{\xi_1}\cdots u_n^{m_n}}{\xi_1!\cdots\xi_n!}, \tag{14}$$

$$F_B^{(n)}\big(a_1\cdots a_n,a_1',\dots,a_n';a_1'';u_1,\dots,u_n\big)=\sum_{\xi_1,\dots,\xi_n=0}^{\infty}\frac{(a_1)_{\xi_1}\cdots(a_n)_{\xi_n}(a_1')_{\xi_1}\cdots(a_n')_{\xi_n}}{(a_1'')_{\xi_1+\cdots+\xi_n}}\frac{u_1^{\xi_1}\cdots u_n^{\xi_n}}{\xi_1!\cdots\xi_n!}, \tag{15}$$

$$F_C^{(n)}\big(a_1,a_1';a_1'',\dots,a_n'';u_1,\dots,u_n\big)=\sum_{\xi_1,\dots,\xi_n=0}^{\infty}\frac{(a_1)_{\xi_1+\cdots+\xi_n}(a_1')_{\xi_1+\cdots+\xi_n}}{(a_1'')_{\xi_1}\cdots(a_n'')_{\xi_n}}\frac{u_1^{\xi_1}\cdots u_n^{\xi_n}}{\xi_1!\cdots\xi_n!}, \tag{16}$$

$$F_D^{(n)}\big(a_1,a_1',\dots,a_n';a_1'';u_1,\dots,u_n\big)=\sum_{\xi_1,\dots,\xi_n=0}^{\infty}\frac{(a_1)_{\xi_1+\cdots+\xi_n}(a_1')_{\xi_1}\cdots(a_n')_{\xi_n}}{(a_1'')_{\xi_1+\cdots+\xi_n}}\frac{u_1^{\xi_1}\cdots u_n^{\xi_n}}{\xi_1!\cdots\xi_n!}. \tag{17}$$

For the number of variables $n=2$, Lauricella functions reduce to Appell series $F_2$, $F_3$, $F_4$, and $F_1$ respectively, and for $n=1$, these functions reduce to the Gauss hypergeometric function ${}_2F_1(\cdot)$.

The Lauricella functions converge as per the following conditions:
- $F_A^{(n)}$ converges when $|u_1|+\cdots+|u_n|<1$;
- $F_B^{(n)}$ converges when $|u_1|<1,\dots,|u_n|<1$;
- $F_C^{(n)}$ converges when $|\sqrt{u_1}|+\cdots+|\sqrt{u_n}|<1$;
- $F_D^{(n)}$ converges when $|u_1|<1,\dots,|u_n|<1$.

Fractional calculus has many applications in the areas of science and engineering like fluid flow, electrical networks, and many others.

The classical Riemann–Liouville fractional integral of order $\beta\in\mathbb{C}$ with $\Re(\beta)>0$ of a function $g$ is given by [5]

$$\big[I_y^{\beta}g\big](y)=\frac{1}{\Gamma(\beta)}\int_0^y g(t)(y-t)^{\beta-1}\,dt,\quad y>0. \tag{18}$$

The classical Riemann–Liouville fractional derivative of order $\beta \in \mathbb{C}$ with $\Re(\beta) < 0$ of a function $g$ is given by

$$[D_y^\beta g](y) = \frac{1}{\Gamma(-\beta)} \int_0^y g(t)(y-t)^{-\beta-1}\,dt, \quad y > 0, \Re(\beta) < 0. \tag{19}$$

## 2 Extension of beta function

Many authors have studied various extensions and generalizations of beta function and hypergeometric functions (see, e.g., [6–10]). In this section, we have made efforts to define the extension of beta function using the Appell series and the Lauricella function.

**Definition 2.1** The extensions of beta function using Appell series (10)–(13), respectively, are defined as follows:

$$\text{(a)} \quad B_{p,q}^{F_1}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{20}$$

where $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3) > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

$$\text{(b)} \quad B_{p,q}^{F_2}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_2\left(a_1,a_2,a_2';a_3,a_3';\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{21}$$

where $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3),\Re(a_3') > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

$$\text{(c)} \quad B_{p,q}^{F_3}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_3\left(a_1,a_1',a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{22}$$

where $\Re(a_1),\Re(a_1'),\Re(a_2),\Re(a_2'),\Re(a_3) > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

$$\text{(d)} \quad B_{p,q}^{F_4}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_4\left(a_1,a_2;a_3,a_3';\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{23}$$

where $\Re(a_1),\Re(a_2),\Re(a_3),\Re(a_3') > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

*Remark* 2.1 When $q = 0$, Appell's double hypergeometric functions reduce to the hypergeometric function

$$\begin{aligned} F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right) &= F_2\left(a_1,a_2,a_2';a_3,a_3';\frac{p}{t^r},0\right) \\ &= F_3\left(a_1,a_1',a_2,a_2';a_3;\frac{p}{t^r},0\right) = F_4\left(a_1,a_2;a_3,a_3';\frac{p}{t^r},0\right) \\ &= {}_2F_1\left(a_1,a_2;a_3;\frac{p}{t^r}\right), \end{aligned}$$

hence for $q = 0$ in Definition 2.1, we obtain the following result:

The extension of beta function using hypergeometric series is defined as follows:

$$B_p(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} {}_2F_1\left(a_1,a_2;a_3;\frac{p}{t^r}\right)dt, \tag{24}$$

where $\Re(a_1),\Re(a_2),\Re(a_3) > 0$, $\Re(p) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

Subsequently, for $p = 1$, $r = 0$, and $a_1, a_2, a_3 = 1$, equations (20)–(23) reduce to the classical beta function $B(\Psi_1, \Psi_2)$, equation (1).

**Definition 2.2** The extensions of beta function using Lauricella series (14)–(17) respectively, are defined as follows:

(a)
$$B^{F_A^{(n)}}_{p_1,\ldots,p_n}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_A^{(n)}\left(a_1,a_1',\ldots,a_n';a_1'',\ldots,a_n'';\frac{p_1}{t^r},\ldots,\frac{p_n}{t^r}\right)dt, \tag{25}$$

where $\Re(a_1),\Re(a_1'),\ldots,\Re(a_n'),\Re(a_1''),\ldots,\Re(a_n'') > 0$, $\Re(p_1),\ldots,\Re(p_n) \ge 0$, $r \ge 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

(b)
$$B^{F_B^{(n)}}_{p_1,\ldots,p_n}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_B^{(n)}\left(a_1,\ldots,a_n,a_1',\ldots,a_n';a_1'';\frac{p_1}{t^r},\ldots,\frac{p_n}{t^r}\right)dt, \tag{26}$$

where $\Re(a_1),\ldots,\Re(a_n),\Re(a_1'),\ldots,\Re(a_n'),\Re(a_1''),> 0$, $\Re(p_1)\ldots,\Re(p_n) \ge 0$, $r \ge 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

(c)
$$B^{F_C^{(n)}}_{p_1,\ldots,p_n}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_C^{(n)}\left(a_1,a_1';a_1'',\ldots,a_n'';\frac{p_1}{t^r},\ldots,\frac{p_n}{t^r}\right)dt, \tag{27}$$

where $\Re(a_1),\Re(a_1'),\Re(a_1''),\ldots,\Re(a_n'') > 0$, $\Re(p_1),\ldots,\Re(p_n) \ge 0$, $r \ge 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

(d)
$$B^{F_D^{(n)}}_{p_1,\ldots,p_n}(\Psi_1,\Psi_2) = \int_0^1 t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_D^{(n)}\left(a_1,a_1',\ldots,a_n';a_1'';\frac{p_1}{t^r},\ldots,\frac{p_n}{t^r}\right)dt, \tag{28}$$

where $\Re(a_1),\Re(a_1'),\ldots,\Re(a_n'),\Re(a_1'') > 0$, $\Re(p_1),\ldots,\Re(p_n) \ge 0$, $r \ge 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

*Remark* 2.2 For $n = 1$, the Lauricella series reduces to the hypergeometric function

$$\begin{aligned}
&F_A^{(n)}\left(a_1,a_1',\ldots,a_n';a_1'',\ldots,a_n'';\frac{p_1}{t^r};\ldots;\frac{p_n}{t^r}\right)\\
&= F_B^{(n)}\left(a_1,\ldots,a_n,a_1',\ldots,a_n';a_1'';\frac{p_1}{t^r};\ldots;\frac{p_n}{t^r}\right)\\
&= F_C^{(n)}\left(a_1,a_1';a_1'',\ldots,a_n'';\frac{p_1}{t^r};\ldots;\frac{p_n}{t^r}\right) = F_D^{(n)}\left(a_1,a_1',\ldots,a_n';a_1'';\frac{p_1}{t^r};\ldots;\frac{p_n}{t^r}\right)\\
&= {}_2F_1\left(a_1,a_1';a_1'';\frac{p_1}{t^r}\right),
\end{aligned}$$

and hence, for $n = 1$ in Definition 2.2, we obtain result (24).

Subsequently, if $p_1 = 1$, $r = 0$, and $a_1, a_1', a_1'' = 1$, equations (25)–(28) reduce to the classical beta function $B(\Psi_1, \Psi_2)$, equation (1).

## 3 Important properties of the extended beta function

In this section, recurrence relations and integral representations have been derived for the new extended beta function.

**Theorem 3.1** *The extension of beta function involving Appell series $F_1(\cdot)$ satisfies the following*:

$$B_{p,q}^{F_1}(\Psi_1, \Psi_2) = B_{p,q}^{F_1}(\Psi_1 + 1, \Psi_2) + B_{p,q}^{F_1}(\Psi_1, \Psi_2 + 1). \tag{29}$$

*Proof*

$$\begin{aligned}
\text{RHS} &= B_{p,q}^{F_1}(\Psi_1 + 1, \Psi_2) + B_{p,q}^{F_1}(\Psi_1, \Psi_2 + 1) \\
&= \int_0^1 t^{\Psi_1}(1-t)^{\Psi_2 - 1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt \\
&\quad + \int_0^1 t^{\Psi_1 - 1}(1-t)^{\Psi_2} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt \\
&= \int_0^1 t^{\Psi_1 - 1}(1-t)^{\Psi_2 - 1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt \\
&= B_{p,q}^{F_1}(\Psi_1, \Psi_2) = \text{LHS}.
\end{aligned}$$

This proves the desired result (29). □

**Theorem 3.2** *The extension of beta function involving Appell series $F_1(\cdot)$ satisfies the following*:

$$B_{p,q}^{F_1}(\Psi_1, 1 - \Psi_2) = \sum_{n=0}^{\infty} \frac{(\Psi_2)_n}{n!} B_{p,q}^{F_1}(\Psi_1 + n, 1). \tag{30}$$

*Proof*

$$\text{LHS} = B_{p,q}^{F_1}(\Psi_1, 1 - \Psi_2) = \int_0^1 t^{\Psi_1 - 1}(1-t)^{-\Psi_2} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt.$$

$$\text{Using } (1-t)^{-\Psi_2} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\Psi_2)_n, \quad |t| < 1,$$

$$B_{p,q}^{F_1}(\Psi_1, 1 - \Psi_2) = \int_0^1 t^{\Psi_1 - 1} \sum_{n=0}^{\infty} \frac{t^n}{n!}(\Psi_2)_n F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt.$$

Interchanging the order of integration and summation, we get

$$B_{p,q}^{F_1}(\Psi_1, 1 - \Psi_2) = \sum_{n=0}^{\infty} \frac{(\Psi_2)_n}{n!} \int_0^1 t^{\Psi_1 + n - 1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt$$

$$= \sum_{n=0}^{\infty} \frac{(\Psi_2)_n}{n!} B_{p,q}^{F_1}(\Psi_1 + n, 1) = \text{RHS}.$$

This proves the desired result (30). □

**Theorem 3.3** *The following result for the extension of beta function involving Appell series $F_1(\cdot)$ holds true*:

$$B_{p,q}^{F_1}(\Psi_1, \Psi_2) = \sum_{n=0}^{\infty} B_{p,q}^{F_1}(\Psi_1 + n, \Psi_2 + 1). \tag{31}$$

*Proof*

$$\text{LHS} = B_{p,q}^{F_1}(\Psi_1, \Psi_2) = \int_0^1 t^{\Psi_1 - 1}(1-t)^{\Psi_2 - 1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt.$$

Using $(1-t)^{\Psi_2 - 1} = (1-t)^{\Psi_2} \sum_{n=0}^{\infty} t^n$,

$$B_{p,q}^{F_1}(\Psi_1, \Psi_2) = \int_0^1 t^{\Psi_1 - 1}(1-t)^{\Psi_2} \sum_{n=0}^{\infty} t^n F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt.$$

Interchanging the order of integration and summation, we obtain

$$\begin{aligned} B_{p,q}^{F_1}(\Psi_1, \Psi_2) &= \sum_{n=0}^{\infty} \int_0^1 t^{\Psi_1 + n - 1}(1-t)^{\Psi_2} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{t^r}, \frac{q}{(1-t)^r}\right) dt \\ &= \sum_{n=0}^{\infty} B_{p,q}^{F_1}(\Psi_1 + n, \Psi_2 + 1) = \text{RHS}. \end{aligned}$$

This proves the desired result (31). □

**Theorem 3.4** *The following result for the extension of beta function involving Appell series $F_1(\cdot)$ holds true*:

$$B_{p,q}^{F_1}(d, -d-n) = \sum_{s=0}^{n} {}^nC_s B_{p,q}^{F_1}(d+s, -d-s), \tag{32}$$

*where* ${}^nC_s = \frac{n!}{s!(n-s)!}$.

*Proof* From equation (29), we have

$$B_{p,q}^{F_1}(\Psi_1, \Psi_2) = B_{p,q}^{F_1}(\Psi_1 + 1, \Psi_2) + B_{p,q}^{F_1}(\Psi_1, \Psi_2 + 1).$$

Let $\Psi_1 = d$ and $\Psi_2 = -d - n$, then

$$B_{p,q}^{F_1}(d, -d-n) = B_{p,q}^{F_1}(d+1, -d-n) + B_{p,q}^{F_1}(d, -d-n+1). \tag{33}$$

Substituting $n = 1, 2, 3, \ldots$ in equation (33), we get

$$B_{p,q}^{F_1}(d, -d-1) = B_{p,q}^{F_1}(d+1, -d-1) + B_{p,q}^{F_1}(d, -d),$$

$$B_{p,q}^{F_1}(d,-d-2) = B_{p,q}^{F_1}(d+2,-d-2) + 2B_{p,q}^{F_1}(d+1,-d-1) + B_{p,q}^{F_1}(d,-d),$$
$$B_{p,q}^{F_1}(d,-d-3) = B_{p,q}^{F_1}(d+3,-d-3) + 3B_{p,q}^{F_1}(d+2,-d-2) + 3B_{p,q}^{F_1}(d+1,-d-1) + B_{p,q}^{F_1}(d,-d),$$

and so on.

On generalizing, we get our desired result (32). □

**Theorem 3.5** (Integral representation) *The following integral representations hold true*:

(i)
$$B_{p,q}^{F_1}(\Psi_1,\Psi_2) = 2\int_0^{\frac{\pi}{2}} \cos^{2\Psi_1-1}\theta \sin^{2\Psi_2-1}\theta \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\cos^{2r}\theta},\frac{q}{\sin^{2r}\theta}\right)d\theta, \tag{34}$$

(ii)
$$B_{p,q}^{F_1}(\Psi_1,\Psi_2) = \int_0^{\infty} \frac{u^{\Psi_1-1}}{(1+u)^{\Psi_1+\Psi_2}} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(1+u)^r}{u^r},q(1+u)^r\right)du, \tag{35}$$

(iii)
$$B_{p,q}^{F_1}(\Psi_1,\Psi_2) = 2^{1-\Psi_1-\Psi_2}\int_{-1}^{1}(1+u)^{\Psi_1-1}(1-u)^{\Psi_2-1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{2^r p}{(1+u)^r},\frac{2^r q}{(1-u)^r}\right)du, \tag{36}$$

(iv)
$$B_{p,q}^{F_1}(\Psi_1,\Psi_2) = (c-z)^{1-\Psi_1-\Psi_2}\int_z^c (u-z)^{\Psi_1-1}(c-u)^{\Psi_2-1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(c-z)^r}{(u-z)^r},\frac{q(c-z)^r}{(c-u)^r}\right)du, \tag{37}$$

(v)
$$B_{p,q}^{F_1}(\Psi_1,\Psi_2) = \int_0^{\frac{\pi}{4}} \tanh^{2\Psi_1-2}\theta \,\mathrm{sech}^{2\Psi_2}\theta \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\tanh^{2r}\theta},\frac{q}{\mathrm{sech}^{2r}\theta}\right)d\theta, \tag{38}$$

*where* $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3) > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, *and* $\Re(\Psi_2) > 0$.

*Proof* Substitute $t = \cos^2\theta$, $t = \frac{u}{(1+u)}$, $t = \frac{(1+u)}{2}$, $t = \frac{(u-z)}{(c-z)}$, and $t = \tanh^2\theta$ in equation (20) to get equations (34)–(38), respectively. □

Similarly, we can prove the above results for $B_{p,q}^{F_2}(\Psi_1,\Psi_2)$, $B_{p,q}^{F_3}(\Psi_1,\Psi_2)$, and $B_{p,q}^{F_4}(\Psi_1,\Psi_2)$.

## 4 Statistical distribution involving extended beta function

In this section, application of the newly defined extension of beta function in statistics has been discussed. We define the extended beta distribution and derive the results for its mean, variance, moment generating function and cumulative distribution function.

**Definition 4.1** Distribution of a new extended beta function involving Appell function $F_1(\cdot)$ is defined by

$$f(t) = \begin{cases} \frac{1}{B_{p,q}^{F_1}(\Psi_1,\Psi_2)} t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_1(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}), & 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{39}$$

where $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3) > 0$, $\Re(p),\Re(q) \geq 0$, $r \geq 0$, $\Re(\Psi_1) > 0$, and $\Re(\Psi_2) > 0$.

If $\eta$ is any real number, then the mean of the extended beta distribution defined above is given by

$$E(Y^n) = \int_0^1 Y^n f(Y)\,dt = \frac{B_{p,q}^{F_1}(\Psi_1+\eta,\Psi_2)}{B_{p,q}^{F_1}(\Psi_1,\Psi_2)}, \tag{40}$$

where $Y$ is any random variable.

If $\eta = 1$, then the mean becomes

$$E(Y) = \frac{B_{p,q}^{F_1}(\Psi_1+1,\Psi_2)}{B_{p,q}^{F_1}(\Psi_1,\Psi_2)}.$$

Variance of the distribution is given by

$$\sigma^2 = E(Y^2) - E(Y)^2 = \frac{B_{p,q}^{F_1}(\Psi_1,\Psi_2)B_{p,q}^{F_1}(\Psi_1+2,\Psi_2) - \{B_{p,q}^{F_1}(\Psi_1+1,\Psi_2)\}^2}{\{B_{p,q}^{F_1}(\Psi_1,\Psi_2)\}^2}. \tag{41}$$

Moment generating function of the distribution is

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E(Y^n) = \frac{1}{B_{p,q}^{F_1}(\Psi_1,\Psi_2)} \sum_{n=0}^{\infty} B_{p,q}^{F_1}(\Psi_1+n,\Psi_2)\frac{t^n}{n!}. \tag{42}$$

Cumulative distribution function is given as

$$F(y) = \frac{B_{y,p,q}^{F_1}(\Psi_1,\Psi_2)}{B_{p,q}^{F_1}(\Psi_1,\Psi_2)}, \tag{43}$$

with

$$B_{y,p,q}^{F_1}(\Psi_1,\Psi_2) = \int_0^y t^{\Psi_1-1}(1-t)^{\Psi_2-1} F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt.$$

Similarly, we can obtain the above results for $B_{p,q}^{F_2}(\Psi_1,\Psi_2)$, $B_{p,q}^{F_3}(\Psi_1,\Psi_2)$, and $B_{p,q}^{F_4}(\Psi_1,\Psi_2)$.

## 5 Extension of hypergeometric and confluent hypergeometric functions using extended beta function

Here, we define a new extension of hypergeometric and confluent hypergeometric functions using the new extension of beta function.

**Definition 5.1** The extension of hypergeometric function using the newly defined beta function involving Appell series $F_1(\cdot)$ is

$$F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)=\sum_{n=0}^{\infty}(\Psi_1)_n\frac{B_{p,q}^{F_1}(\Psi_2+n,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^n}{n!}, \tag{44}$$

with $\Re(\Psi_3)>\Re(\Psi_2)>0$ and $\Re(p),\Re(q)\geq 0$ and $|w|<1$.

**Definition 5.2** The extension of confluent hypergeometric function using the newly defined beta function involving Appell series $F_1(\cdot)$ is

$$\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w)=\sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_2+n,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^n}{n!}, \tag{45}$$

with $\Re(\Psi_3)>\Re(\Psi_2)>0$ and $\Re(p),\Re(q)\geq 0$ and $|w|<1$.

Similarly, we can define $F_{p,q}^{F_2}(\Psi_1,\Psi_2;\Psi_3;w)$, $F_{p,q}^{F_3}(\Psi_1,\Psi_2;\Psi_3;w)$, $F_{p,q}^{F_4}(\Psi_1,\Psi_2;\Psi_3;w)$, $\Phi_{p,q}^{F_2}(\Psi_2;\Psi_3;w)$, $\Phi_{p,q}^{F_3}(\Psi_2;\Psi_3;w)$, and $\Phi_{p,q}^{F_4}(\Psi_2;\Psi_3;w)$.

*Remark* 5.1 When $q=0$ and subsequently if $p=1$, $r=0$, $a_1,a_2,a_3=1$, equations (44)–(45) reduce to the Gauss hypergeometric function (2) and confluent hypergeometric function (3), respectively.

### 5.1 Integral representation

**Theorem 5.1** *The extended hypergeometric function has the following integral representation*:

$$\begin{aligned}F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)&=\frac{1}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^1 t^{\Psi_2-1}(1-t)^{\Psi_3-\Psi_2-1}(1-tw)^{-\Psi_1}\\&\quad\times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt,\end{aligned} \tag{46}$$

*where* $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$, $\Re(\Psi_3)>\Re(\Psi_2)>0$, $|w|<1$, *and* $|\arg(1-t)|<\pi$.

*Proof* By definition (44),

$$F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)=\sum_{n=0}^{\infty}(\Psi_1)_n\frac{B_{p,q}^{F_1}(\Psi_2+n,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^n}{n!}.$$

By definition (20) of the extended beta function

$$\begin{aligned}F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)&=\frac{1}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^1 t^{\Psi_2+n-1}(1-t)^{\Psi_3-\Psi_2-1}\sum_{n=0}^{\infty}(\Psi_1)_n\\&\quad\times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt\frac{w^n}{n!}.\end{aligned}$$

Using $\sum_{n=0}^{\infty}\frac{(\Psi_1)_n(tw)^n}{n!}=(1-t)^{-\Psi_1}$, we get the desired result (46). □

**Theorem 5.2** *The extended confluent hypergeometric function has the following integral representation*:

$$\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^1 t^{\Psi_2-1}(1-t)^{\Psi_3-\Psi_2-1}\exp(wt) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{47}$$

$$\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{\exp(w)}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^1 t^{\Psi_2-1}(1-t)^{\Psi_3-\Psi_2-1}\exp(-wt) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt, \tag{48}$$

*where* $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$, $\Re(\Psi_3)>\Re(\Psi_2)>0$, *and* $|w|<1$.

*Proof* By definition (45)

$$\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \sum_{n=0}^{\infty}\frac{B_{p,q}^{F_1}(\Psi_2+n,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^n}{n!}.$$

By definition (20) of the extended beta function

$$\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)}\sum_{n=0}^{\infty}\int_0^1 t^{\Psi_2+n-1}(1-t)^{\Psi_3-\Psi_2-1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{t^r},\frac{q}{(1-t)^r}\right)dt\frac{w^n}{n!}.$$

Using $\sum_{n=0}^{\infty}\frac{(tw)^n}{n!} = \exp(wt)$, we get the desired result (47).

Replacing $t$ by $(1-t)$ in equation (47), we get result (48). □

**Theorem 5.3** *The following integral representations for the extended hypergeometric function hold true*:

(i) $$F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^{\infty} a^{\Psi_2-1}(1+a)^{\Psi_1-\Psi_3}\bigl(1+a(1-w)\bigr)^{-\Psi_1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(1+a)^r}{a^r},q(1+a)^r\right)da, \tag{49}$$

(ii) $$F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w) = \frac{2}{B(\Psi_2,\Psi_3-\Psi_2)}\int_0^{\frac{\pi}{2}} \cos^{2\Psi_2-1}\theta\sin^{2\Psi_3-2\Psi_2-1}\theta \times \left(1-\cos^2\theta w\right)^{-\Psi_1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\cos^{2r}\theta},\frac{q}{\sin^{2r}\theta}\right)d\theta, \tag{50}$$

$$\text{(iii)}\quad F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w) = \frac{2^{1+\Psi_1-\Psi_3}}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{-1}^{1} (1-a)^{\Psi_3-\Psi_2-1}\big(2-w(1+a)\big)^{-\Psi_1} \times (1+a)^{\Psi_2-1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{2^r p}{(1+a)^r},\frac{2^r q}{(1-a)^r}\right) da, \tag{51}$$

$$\text{(iv)}\quad F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w) = \frac{(c-u)^{1+\Psi_1-\Psi_3}}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{u}^{c} (a-u)^{\Psi_2-1}\big((c-u)-w(a-u)\big)^{-\Psi_1} \times (c-a)^{\Psi_3-\Psi_2-1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(c-u)^r}{(a-u)^r},\frac{q(c-u)^r}{(c-a)^r}\right) da, \tag{52}$$

$$\text{(v)}\quad F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{0}^{\frac{\pi}{4}} \tanh^{2\Psi_2-2}\theta \operatorname{sech}^{2\Psi_3-2\Psi_2}\theta \times \big(1-\tanh^2\theta w\big)^{-\Psi_1} \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\tanh^{2r}\theta},\frac{q}{\operatorname{sech}^{2r}\theta}\right) d\theta. \tag{53}$$

*Proof* Substitute $t=\frac{a}{1+a}$, $t=\cos^2\theta$, $t=\frac{1+a}{2}$, $t=\frac{a-u}{c-u}$, and $t=\tanh^2\theta$ in equation (46) to get equations (49)–(53). □

**Theorem 5.4** *The following integral representations for the extended confluent hypergeometric function hold true:*

$$\text{(i)}\quad \Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{0}^{\infty} a^{\Psi_2-1}(1+a)^{-\Psi_3} \exp\left(\frac{wa}{1+a}\right) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(1+a)^r}{a^r},q(1+a)^r\right) da, \tag{54}$$

$$\text{(ii)}\quad \Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{2}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{0}^{\frac{\pi}{2}} \cos^{2\Psi_2-1}\theta \sin^{2\Psi_3-2\Psi_2-1}\theta \exp\left(w\cos^2\theta\right) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\cos^{2r}\theta},\frac{q}{\sin^{2r}\theta}\right) d\theta, \tag{55}$$

$$\text{(iii)}\quad \Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{2^{1-\Psi_3}}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{-1}^{1} (1+a)^{\Psi_2-1}(1-a)^{\Psi_3-\Psi_2-1} \exp\left(\frac{w(1+a)}{2}\right) \times F_1\left(a_1,a_2,a_2';a_3;\frac{2^r p}{(1+a)^r},\frac{2^r q}{(1-a)^r}\right) da, \tag{56}$$

$$\text{(iv)}\quad \Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{(c-u)^{1-\Psi_3}}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{u}^{c} (a-u)^{\Psi_2-1}(c-a)^{\Psi_3-\Psi_2-1} \exp\left(\frac{w(a-u)}{(c-u)}\right) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p(c-u)^r}{(a-u)^r},\frac{q(c-u)^r}{(c-a)^r}\right) da, \tag{57}$$

$$\text{(v)}\quad \Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w) = \frac{1}{B(\Psi_2,\Psi_3-\Psi_2)} \int_{0}^{\frac{\pi}{4}} \tanh^{2\Psi_2-2}\theta \operatorname{sech}^{2\Psi_3-2\Psi_2}\theta \exp\left(w\tanh^2\theta\right) \times F_1\left(a_1,a_2,a_2';a_3;\frac{p}{\tanh^{2r}\theta},\frac{q}{\operatorname{sech}^{2r}\theta}\right) d\theta. \tag{58}$$

*Proof* Substitute $t = \frac{a}{1+a}$, $t = \cos^2\theta$, $t = \frac{1+a}{2}$, $t = \frac{a-u}{c-u}$, and $t = \tanh^2\theta$ in equation (47) to get equations (54)–(58). □

### 5.2 Differentiation formula

**Theorem 5.5** *The following differentiation formulas for the extended hypergeometric and confluent hypergeometric function hold true*:

$$\frac{d^n}{dw^n}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \frac{(\Psi_1)_n(\Psi_2)_n}{(\Psi_3)_n}F_{p,q}^{F_1}(\Psi_1+n,\Psi_2+n;\Psi_3+n;w) \tag{59}$$

*and*

$$\frac{d^n}{dw^n}\left\{\Phi_{p,q}^{F_1}(\Psi_2;\Psi_3;w)\right\} = \frac{(\Psi_2)_n}{(\Psi_3)_n}\Phi_{p,q}^{F_1}(\Psi_2+n;\Psi_3+n;w). \tag{60}$$

*Proof* Differentiating equation (44) with respect to $w$, we get

$$\frac{d}{dw}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \sum_{n=1}^{\infty}(\Psi_1)_n\frac{B_{p,q}^{F_1}(\Psi_2+n,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^{n-1}}{(n-1)!}.$$

Replacing $n$ by $n+1$

$$\frac{d}{dw}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \sum_{n=0}^{\infty}(\Psi_1)_{n+1}\frac{B_{p,q}^{F_1}(\Psi_2+n+1,\Psi_3-\Psi_2)}{B(\Psi_2,\Psi_3-\Psi_2)}\frac{w^{n}}{n!}. \tag{61}$$

Using $B(h,k-h) = \frac{k}{h}B(h+1,k-h)$ in equation (61), we get

$$\frac{d}{dw}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \sum_{n=0}^{\infty}(\Psi_1)_{n+1}\left(\frac{\Psi_2}{\Psi_3}\right)\frac{B_{p,q}^{F_1}(\Psi_2+n+1,\Psi_3-\Psi_2)}{B(\Psi_2+1,\Psi_3-\Psi_2)}\frac{w^{n}}{n!} \tag{62}$$

$$\implies \frac{d}{dw}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \sum_{n=0}^{\infty}(\Psi_1)_{n}\left(\frac{\Psi_1\Psi_2}{\Psi_3}\right)\frac{B_{p,q}^{F_1}(\Psi_2+n+1,\Psi_3-\Psi_2)}{B(\Psi_2+1,\Psi_3-\Psi_2)}\frac{w^{n}}{n!}$$
$$= \frac{\Psi_1\Psi_2}{\Psi_3}F_{p,q}^{F_1}(\Psi_1+1,\Psi_2+1;\Psi_3+1;w).$$

Again differentiating equation (62) with respect to $w$, we get

$$\frac{d^2}{dw^2}\left\{F_{p,q}^{F_1}(\Psi_1,\Psi_2;\Psi_3;w)\right\} = \frac{\Psi_1(\Psi_1+1)\Psi_2(\Psi_2+1)}{\Psi_3(\Psi_3+1)}F_{p,q}^{F_1}(\Psi_1+2,\Psi_2+2;\Psi_3+2;w).$$

Continuing like this, $n$ times, we get the desired result (59).

We can prove result (60) in a similar way. □

Similarly, we can prove the above results for $F_{p,q}^{F_2}(\Psi_1,\Psi_2;\Psi_3;w)$, $F_{p,q}^{F_3}(\Psi_1,\Psi_2;\Psi_3;w)$, $F_{p,q}^{F_4}(\Psi_1,\Psi_2;\Psi_3;w)$, $\Phi_{p,q}^{F_2}(\Psi_2;\Psi_3;w)$, $\Phi_{p,q}^{F_3}(\Psi_2;\Psi_3;w)$, and $\Phi_{p,q}^{F_4}(\Psi_2;\Psi_3;w)$.

## 6 Extension of Riemann–Liouville fractional operators

In this section, we extend the Riemann–Liouville fractional operators using the Appell series and derive its properties using the new extension of beta function.

**Definition 6.1** The extended Riemann–Liouville fractional integral is defined as follows:

$$I_y^\beta\big[g(y):p,q\big] = \frac{1}{\Gamma(\beta)}\int_0^y g(t)(y-t)^{\beta-1}F_1\left(a_1,a_2,a_2';a_3;\frac{py^r}{t^r},\frac{qy^r}{(y-t)^r}\right)dt, \tag{63}$$

where $\Re(\beta)>0$, $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$.

**Definition 6.2** The extended Riemann–Liouville fractional derivative is defined as follows:

$$D_y^\beta\big[g(y):p,q\big] = \frac{1}{\Gamma(-\beta)}\int_0^y g(t)(y-t)^{-\beta-1}F_1\left(a_1,a_2,a_2';a_3;\frac{py^r}{t^r},\frac{qy^r}{(y-t)^r}\right)dt, \tag{64}$$

where $\Re(\beta)<0$, $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$.

Similarly, we can define the extended Riemann–Liouville fractional operators using Appell series $F_2(\cdot)$, $F_3(\cdot)$, and $F_4(\cdot)$.

**Theorem 6.1** *If* $\Re(\beta)>0$, $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$, *then*

$$I_y^\beta\big[g(y)=y^\eta:p,q\big] = \frac{1}{\Gamma(\beta)}B_{p,q}^{F_1}(\eta+1,\beta)y^{\eta+\beta}. \tag{65}$$

*Proof* From Definition 6.1

$$I_y^\beta\big[g(y)=y^\eta:p,q\big] = \frac{1}{\Gamma(\beta)}\int_0^y t^\eta(y-t)^{\beta-1}F_1\left(a_1,a_2,a_2';a_3;\frac{py^r}{t^r},\frac{qy^r}{(y-t)^r}\right)dt. \tag{66}$$

Let $t=yu$, then equation (66) becomes

$$I_y^\beta\big[g(y)=y^\eta:p,q\big] = \frac{1}{\Gamma(\beta)}y^{\eta+\beta}\int_0^1 u^\eta(1-u)^{\beta-1}F_1\left(a_1,a_2,a_2';a_3;\frac{p}{u^r},\frac{q}{(1-u)^r}\right)du. \tag{67}$$

Using equation (20) in (67), we get our desired result (65). □

**Theorem 6.2** *If* $\Re(\beta)>0$, $\Re(a_1),\Re(a_2),\Re(a_2'),\Re(a_3)>0$, $\Re(p),\Re(q)\geq 0$, $r\geq 0$, *and* $g(t)=\sum_{m=0}^\infty b_m t^m$, $|t|<1$, *then*

$$\text{(i)}\quad I_y^\beta\big[g(y):p,q\big] = \frac{1}{\Gamma(\beta)}\sum_{m=0}^\infty b_m B_{p,q}^{F_1}(m+1,\beta)y^{m+\beta}. \tag{68}$$

$$\text{(ii)}\quad I_y^\beta\big[t^{\lambda-1}g(y):p,q\big] = \frac{y^{\lambda+\beta-1}}{\Gamma(\beta)}\sum_{m=0}^\infty b_m B_{p,q}^{F_1}(m+\lambda,\beta)y^{m}. \tag{69}$$

*Proof* (i) From Definition 6.1

$$I_y^\beta\big[g(y):p,q\big] = \frac{1}{\Gamma(\beta)}\int_0^y \sum_{m=0}^\infty b_m t^m(y-t)^{\beta-1}F_1\left(a_1,a_2,a_2';a_3;\frac{py^r}{t^r},\frac{qy^r}{(y-t)^r}\right)dt.$$

Interchanging the order of integration and summation, we get

$$I_y^{\beta}\left[g(y):p,q\right] = \frac{1}{\Gamma(\beta)}\sum_{m=0}^{\infty} b_m \int_0^y t^m (y-t)^{\beta-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{py^r}{t^r}, \frac{qy^r}{(y-t)^r}\right) dt. \tag{70}$$

Using equation (65) in (70), we get our desired result (68).

(ii) From Definition 6.1, we have

$$I_y^{\beta}\left[t^{\lambda-1}g(y):p,q\right] = \frac{1}{\Gamma(\beta)}\int_0^y \sum_{m=0}^{\infty} b_m t^{m+\lambda-1} (y-t)^{\beta-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{py^r}{t^r}, \frac{qy^r}{(y-t)^r}\right) dt.$$

Interchanging the order of integration and summation, we get

$$\begin{aligned} &I_y^{\beta}\left[g(y):p,q\right] \\ &\quad = \frac{1}{\Gamma(\beta)}\sum_{m=0}^{\infty} b_m \int_0^y t^{m+\lambda-1} (y-t)^{\beta-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{py^r}{t^r}, \frac{qy^r}{(y-t)^r}\right) dt. \end{aligned} \tag{71}$$

Using equation (65) in (71), we get our desired result (69). □

**Theorem 6.3** *If* $\Re(\mu-\eta) > 0$, $\Re(a_1), \Re(a_2), \Re(a_2'), \Re(a_3) > 0$, $\Re(p), \Re(q) \geq 0$, $r \geq 0$, *then*

$$I_y^{\mu-\eta}\left[y^{\eta-1}(1-y)^{-\beta}:p,q\right] = \frac{\Gamma(\eta)}{\Gamma(\mu)} y^{\mu-1} F_{p,q}^{F_1}(\beta, \eta; \mu; y). \tag{72}$$

*Proof* From Definition 6.1

$$\begin{aligned} &I_y^{\mu-\eta}\left[y^{\eta-1}(1-y)^{-\beta}:p,q\right] \\ &\quad = \frac{1}{\Gamma(\mu-\eta)}\int_0^y t^{\eta-1}(1-t)^{-\beta}(y-t)^{\mu-\eta-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{py^r}{t^r}, \frac{qy^r}{(y-t)^r}\right) dt. \end{aligned} \tag{73}$$

Let $t = yu$, then equation (73) becomes

$$\begin{aligned} &I_y^{\mu-\eta}\left[y^{\eta-1}(1-y)^{-\beta}:p,q\right] \\ &\quad = \frac{y^{\mu-1}}{\Gamma(\mu-\eta)}\int_0^1 u^{\eta-1}(1-yu)^{-\beta}(1-u)^{\mu-\eta-1} F_1\left(a_1, a_2, a_2'; a_3; \frac{p}{u^r}, \frac{q}{(1-u)^r}\right) du. \end{aligned} \tag{74}$$

Using equation (46) in (74), we get

$$I_y^{\mu-\eta}\left[y^{\eta-1}(1-y)^{-\beta}:p,q\right] = \frac{y^{\mu-1}}{\Gamma(\mu-\eta)} B(\eta, \mu-\eta) F_{p,q}^{F_1}(\beta, \eta; \mu; y).$$

On simplification, we get our desired result (72). □

**Theorem 6.4** *If* $\Re(\beta) < 0$, $\Re(a_1), \Re(a_2), \Re(a_2'), \Re(a_3) > 0$, $\Re(p), \Re(q) \geq 0$, $r \geq 0$, *then*

$$D_y^{\beta}\left[g(y) = y^{\eta}:p,q\right] = \frac{1}{\Gamma(-\beta)} B_{p,q}^{F_1}(\eta+1, -\beta) y^{\eta-\beta}. \tag{75}$$

*Proof* Using Definition 6.2 and following the same method as in Theorem 6.1, we get our desired result (75). □

**Theorem 6.5** *If* $\Re(\beta) < 0$, $\Re(a_1), \Re(a_2), \Re(a_2'), \Re(a_3) > 0$, $\Re(p), \Re(q) \geq 0$, $r \geq 0$, *and* $g(t) = \sum_{m=0}^{\infty} b_m t^m$, $|t| < 1$, *then*

$$\text{(i)} \quad D_y^{\beta}\left[g(y) : p, q\right] = \frac{1}{\Gamma(-\beta)} \sum_{m=0}^{\infty} b_m B_{p,q}^{F_1}(m+1, -\beta) y^{m-\beta}. \tag{76}$$

$$\text{(ii)} \quad D_y^{\beta}\left[t^{\lambda-1} g(y) : p, q\right] = \frac{y^{\lambda-\beta-1}}{\Gamma(-\beta)} \sum_{m=0}^{\infty} b_m B_{p,q}^{F_1}(m+\lambda, -\beta) y^{m}. \tag{77}$$

*Proof* Using Definition 6.2 and following the same method as in Theorem 6.2, we get our desired results (76) and (77). □

**Theorem 6.6** *If* $\Re(\mu - \eta) < 0$, $\Re(a_1), \Re(a_2), \Re(a_2'), \Re(a_3) > 0$, $\Re(p), \Re(q) \geq 0$, $r \geq 0$, *then*

$$D_y^{\mu-\eta}\left[y^{\mu-1}(1-y)^{-\beta} : p, q\right] = \frac{\Gamma(\mu)}{\Gamma(\eta)} y^{\eta-1} F_{p,q}^{F_1}(\beta, \mu; \eta; y). \tag{78}$$

*Proof* Using Definition 6.2 and following the same method as in Theorem 6.3, we get our desired result (78). □

Similarly, we can derive the above results for the extended Riemann–Liouville fractional operators involving Appell series $F_2(\cdot)$, $F_3(\cdot)$, and $F_4(\cdot)$.

Also, in a similar way, we can prove all the above properties (from Sects. 3–6) for the extension of beta function using Lauricella functions (Definition 2.2).

## 7 Conclusion

In this paper, we have discussed some extensions of beta function, Gauss hypergeometric, and confluent hypergeometric function. A new extension of the classical beta function using Appell series and Lauricella function has been obtained, which reduces to the classical beta function for specific values of the parameters. The integral representations and properties of the newly defined beta function have been evaluated. Further, the statistical distribution using the newly defined beta function has been defined and the mean, variance, moment generating function and cumulative distribution function have been discussed here. Thereafter, the extended beta function has been used to define a new extension of hypergeometric and confluent hypergeometric functions and to discuss their properties like integral representations and differentiation formulas. Moreover, extension of the Riemann–Liouville fractional operators using Appell series and Lauricella function has been defined and its various properties have been discussed using the new extension of beta function. All the results obtained here are reducible to a variety of known results involving classical beta function, Gauss hypergeometric, confluent hypergeometric functions, and many others. In the future, the operators of fractional derivatives, fractional integration, and integral transforms can be applied to the extended beta, hypergeometric, and confluent hypergeometric functions, and several image formulas can be established (see, e.g., [11–13]).

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**Authors' contributions**
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